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Office of Naval Research
Conract Number N00014-85-K-0597

NSF Materials Research Laboratory
Grant DMR-8316893

October 1986

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Abstract

Attention is focussed on the asymptotic stress and deformation fields near the edge of a mode I crack propagating steadily in an elastic-perfectly plastic incompressible material under plane strain conditions. Features of previously reported results on this problem are reviewed, with a view toward establishing discriminating characteristics. An asymptotic solution valid for all crack speeds is constructed. This solution has the properties that the angular variation of stress and particle velocity around the crack tip are continuous and the plastic strains are bounded at the crack tip. The construction depends on the boundedness of hydrostatic stress at the crack tip, and the conditions under which this stress measure is bounded are discussed in an Appendix. The possibility of an asymptotic field with logarithmically singular plastic strain at the crack edge is examined. While fields with this feature can be constructed by admitting discontinuities in stress and particle velocity, it is shown that the sequence of mechanical states experienced in the discontinuities are inconsistent with the principle of maximum plastic work. The dynamic asymptotic field does not reduce to the generally accepted quasi-static asymptotic field in the limit as the crack speed goes to zero. It has been established for the equivalent case of mode III elastic-plastic crack growth that the domain of validity of the dynamic asymptotic field vanishes as the crack speed vanishes, and the implication is that the mode I result has the same property.

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1. Introduction

Steady-state dynamic propagation of a mode I crack in an elastic-perfectly plastic incompressible material under plane strain conditions is considered. The material is assumed to be governed by the Tresca yield condition and the associated flow rule. Aspects of this problem have been considered in the past by Slepyan (1976), Achenbach and Dunayevsky (1981), Gao and Nemat-Nasser (1983) and Lam and Freund (1985). Both Slepyan (1976) and Achenbach and Dunayevsky (1981) took elastic compressibility into account. They were able to extract solutions valid very close to the crack tip, but their solutions were also restricted to vanishing crack speed. It should be noted that in both studies the Tresca yield condition was used together with the Mises flow rule, so that the fields described are consistent with normality of the plastic strain rate to the yield surface only in the limit of incompressibility. In addition, both studies imposed restrictions on the out-of-plane deformation which arise from assuming normality of the plastic strain rate to the Tresca yield surface. This results in volumetric plastic strains which vanish only in the incompressible limit. On the other hand, Gao and Nemat-Nasser (1983) restricted their attention to the fully incompressible case. They reported a solution with jumps in stress and particle velocity along radial lines emanating from the crack tip for all crack speeds between zero and the elastic shear wave speed of the material. Lam and Freund (1985) obtained an extension of the Achenbach and Dunayevsky (1981) solution to crack speeds between zero and the shear wave speed for the case of elastic incompressibility, and examined the case of elastic compressibility numerically.

All previous investigators assumed that the limiting value of hydrostatic stress at the crack tip is finite. This critical assumption can be used to show that the particle velocity in the direction of crack propagation, say v_1, has the form $v_1 \sim A \ln(R/r) + B_1(\theta)$ as $r \to 0$ in terms of polar coordinates r, θ centered at the crack tip. Slepyan (1976) and Achenbach and Dunayevsky (1981) obtained solutions which correspond to the case $A = 0$, while Gao and Nemat-Nasser (1983) looked at the situation $A \neq 0$. A direct comparison of these solutions is possible only for elastic incompressibility and very small crack speed. From a comparison on this basis it is found that the solution of Gao and Nemat-Nasser (1983) is discontinuous, as already noted, and that the plastic strain components are singular at the crack tip, while both the solution

of Slepyan (1976) and that of Achenbach and Dunayevsky (1981) include continuous stress variations and bounded plastic strains. For crack speeds up to 0.1 times the shear wave speed, the computational results of Lam and Freund (1985) are in good agreement with the Achenbach and Dunayevsky (1981) solution. Each of the available asymptotic solutions corresponds to plastic deformation over the full angular range about the crack tip, that is, there is no elastic unloading sector, and in no case does any solution completely reduce to the corresponding quasi-static result in the limit of vanishing crack speed. The computational results of Lam and Freund (1985), however, suggest the presence of some elastic unloading downstream from the crack tip.

The problem is re-examined here in an effort to resolve the discrepancies among the available solutions, or at least to identify a discriminating characteristic. It is shown in the Appendix that, given certain reasonable assumptions concerning the behavior of the stresses and the velocities near the crack tip, the hydrostatic stress must be bounded at the crack tip. As mentioned above (and shown in section 2), this implies that the particle velocity in the direction of crack propagation has a certain mathematical structure. The governing equations that lead to the detailed features of this structure are obtained in section 2. In section 3 the asymptotic solution for the case $A = 0$ is obtained. This solution corresponds to the extension of the Achenbach and Dunayevsky (1981) solution proposed by Lam and Freund (1985).

The case $A \neq 0$ is examined in section 4. It is first shown that a solution of this form satisfying the yield condition over the full angular sector around the crack tip *must be discontinuous*, as argued by Gao and Nemat-Nasser (1983). It is then shown that, for the specific case of elastic incompressibility and perfect-plasticity, the maximum plastic work inequality prohibits the existence of discontinuities in the stress, strain, and particle velocity components. This strong result indicates that the discontinuous solution of Gao and Nemat-Nasser (1983) must be ruled out if all states experienced by a material particle as the crack tip moves by must be consistent with the principle of maximum plastic work. The elimination of discontinuities leaves the introduction of an elastic sector as the only potential means of avoiding a region of negative plastic work. It is shown in section 4, however, that the introduction of such a sector would necessarily violate the unloading condition.

Because the region of negative plastic work cannot be avoided, it must be elim-

inated. Inspection of the governing equations leads to the conclusion that this may be accomplished only by choosing $A = 0$. Thus, the solution presented in section 3 is the only available asymptotic solution consistent with the principle of maximum plastic work. In addition, the solution for an elastic region obtained in section 4 shows that for $A = 0$ any elastic region must be a region of constant (spatially uniform) state. Because the stresses must be continuous at an elastic-plastic boundary, any elastic region must also be at yield. However, regions of this nature have been considered in the solution in section 3, and it is therefore concluded that this solution is unique within the class considered.

2. Governing Equations in Plastically Deforming Sectors

Consider a mode I tensile crack growing steadily at speed u in an elastic-perfectly plastic material under plane strain conditions. The material is assumed to be incompressible, to yield according to the Tresca criterion, and to deform plastically according to the associated flow rule. Both rectangular coordinates x_1, x_2 and polar coordinates r, θ are introduced with their common origin at the tip of the crack. The coordinate systems translate with the crack tip which moves steadily in the x_1-direction with speed u; see Fig. 1. The equations of momentum balance are

$$\frac{\partial \tilde{\sigma}_{ij}}{\partial x_j} = \rho \frac{\partial \tilde{v}_i}{\partial t} , \tag{2.1}$$

where $\tilde{\sigma}_{ij}$ are the rectangular components of the Cauchy stress tensor, $\tilde{v}_i$ are the rectangular components of the particle velocity, and ρ is the mass density of the material. The summation convention for repeated indices is adopted, where Roman indices have the range 1–3. It is convenient to introduce the dimensionless quantities

$$\sigma_{ij} = \frac{1}{\mu}\tilde{\sigma}_{ij} , \qquad v_i = \frac{1}{u}\tilde{v}_i , \qquad m = \frac{u}{c_s} , \tag{2.2}$$

where μ is the elastic shear modulus of the material and $c_s = \sqrt{\mu/\rho}$. Under steady-state conditions the material time derivative of any field quantity f is equivalent to the appropriately scaled spatial derivative

$$\dot{f} = -u \partial f / \partial x_1 . \tag{2.3}$$

The momentum equation (2.1) then becomes

$$\frac{\partial \sigma_{ij}}{\partial x_j} = -m^2 \frac{\partial v_i}{\partial x_1} \ . \tag{2.4}$$

In terms of the dimensionless particle velocity, the rectangular components of the small strain rate tensor are

$$\dot{\epsilon}_{ij} = \frac{u}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \ , \tag{2.5}$$

and additive decomposition of strain into elastic and plastic parts is assumed, that is,

$$\epsilon_{ij} = \epsilon^e_{ij} + \epsilon^p_{ij} \ , \tag{2.6}$$

where ϵ^e_{ij} and ϵ^p_{ij} are the rectangular components of the elastic and plastic strain tensors respectively. The rectangular components of the elastic strain rate, plastic strain rate, and deviatoric stress rate tensors are $\dot{\epsilon}^e_{ij}$, $\dot{\epsilon}^p_{ij}$, and $\dot{s}_{ij}$, respectively. The elastic strain components are related to the dimensionless deviatoric stress components ($s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$) through Hooke's law,

$$\epsilon^e_{ij} = \frac{1}{2} s_{ij} \ . \tag{2.7}$$

The plastic strain rate components are determined from the Prandtl-Reuss flow rule

$$\dot{\epsilon}^p_{ij} = \lambda \mu s_{ij} \ , \tag{2.8}$$

where λ is an unknown non-negative function of position. Combining (2.5–8) yields

$$\frac{u}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) = \lambda \mu s_{ij} + \frac{1}{2}\dot{s}_{ij} \tag{2.9}$$

which describes the material response in plastically deforming regions. The Tresca yield condition in plane strain is

$$\frac{1}{4}(\sigma_{11} - \sigma_{22})^2 + \sigma_{12}^2 = k^2 \tag{2.10}$$

where $k = \sigma_y / 2\mu$ is the normalized yield stress and σ_y is the yield stress in uniaxial tension. This condition states that in plastically deforming regions the maximum shear stress in the plane of deformation has the value $\sigma_y/2$.

The deformation field possesses the symmetry

$$v_1(r,\theta) = v_1(r,-\theta) \quad , \qquad v_2(r,\theta) = -v_2(r,-\theta) \tag{2.11}$$

corresponding to mode I opening. The crack faces are traction free, which is expressed by

$$\sigma_{22}(r,\pm\pi) = 0 \quad , \qquad \sigma_{12}(r,\pm\pi) = 0 \tag{2.12}$$

for all $r > 0$.

As first observed by Koiter (1953) for elastic compressibility, the assumption that the out-of-plane stress σ_{33} is the intermediate principal stress and the subsequent verification from the solution implies that there is no plastic strain in the out-of-plane direction, that is, $\dot\epsilon^p_{33} = 0$. In light of (2.6–7), this implies that

$$s_{11} = -s_{22} \ , \tag{2.13}$$

which allows the yield condition (2.10) to be written as

$$s_{22}^2 + s_{12}^2 = k^2 \ . \tag{2.14}$$

This condition, with (2.13), implies that the deviatoric stresses are bounded for all values of r. If, in addition, it is assumed that the hydrostatic stress $\sigma = \frac{1}{3}\sigma_{kk}$ is bounded, then the momentum equations can be used to discern the form of v_i. (The Appendix contains a discussion of the conditions under which σ may be shown to be bounded.)

A typical momentum equation is

$$\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{12}}{\partial x_2} = -m^2\frac{\partial v_1}{\partial x_1} \ . \tag{2.15}$$

Treating the dependent variables as functions of r and θ, rectangular derivatives may be replaced with polar derivatives by a simple change of variables, for example,

$$\frac{\partial f}{\partial x_1} = \cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial\theta} \ . \tag{2.16}$$

If the stresses are bounded as $r \to 0$, then in view of (2.16), the lefthand side of (2.15) is $O(1/r)$. The right side of (2.15) must therefore be $O(1/r)$. Now (2.16) may be used to determine what form of v_i gives rise to the strongest allowable singularity. If the particle velocity is logarithmically singular as $r \to 0$, then the first term in (2.16) gives rise to a $1/r$ singularity. In order to avoid a stronger singularity arising from the second term in (2.16), the coefficient of the $\ln r$ term must be independent of θ. Alternatively, if the particle velocity is a function of θ as $r \to 0$, then the second term of (2.16) gives rise to a $1/r$ singularity while the first term is $o(1/r)$. In summary, it appears that the two possibilities leading to the strongest allowable singularity in particle acceleration are

$$v_i \sim A_i \ln\left(\frac{R}{r}\right) \,, \quad \frac{dA_i}{d\theta} = 0 \tag{2.17a}$$

or

$$v_i \sim B_i(\theta) \tag{2.17b}$$

where R in (2.17a) is a parameter with the dimensions of length. R is assumed to be the same order of magnitude as the maximum extent of the active plastic zone. The plane strain condition implies that $B_3(\theta) \equiv 0$ and $A_3 \equiv 0$.

It is assumed that the behavior of v_i may change in different angular sectors about the crack tip. Symmetry requires that $A_2 = 0$ in the upstream-most sector, and therefore the result from section 4.2, that the velocity components must be continuous, implies that there can be no logarithmic behavior of v_2 in any angular sector about the crack tip. Thus (2.17b) is the asymptotic representation of v_2 for $0 \leq \theta \leq \pi$, and from here on A_1 is replaced by A.

The incompressibility condition implies that the particle velocity is divergence free, which in plane strain may be expressed by

$$\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} = 0 \,. \tag{2.18}$$

This implies that the velocity components are derivable from a velocity potential $\phi(r,\theta)$ according to

$$v_1 = \frac{\partial \phi}{\partial x_2} \,, \quad v_2 = -\frac{\partial \phi}{\partial x_1} \,. \tag{2.19}$$

If the velocity components are taken to be a combination of the forms indicated in (2.17),

$$v_1 = A \ln\left(\frac{R}{r}\right) + B_1(\theta) \ , \tag{2.20a}$$

$$v_2 = B_2(\theta) \ , \tag{2.20b}$$

then the divergence condition (2.18) requires that A, $B_1(\theta)$, and $B_2(\theta)$ satisfy the relation

$$B_1'(\theta)\tan\theta = B_2'(\theta) - A \ , \tag{2.21}$$

where the prime denotes differentiation with respect to θ. This relation allows integration of (2.19) to obtain

$$\phi(r,\theta) = Ax_2\left(\ln\left(\frac{R}{r}\right) + 1\right) + x_2 B_1(\theta) - x_1 B_2(\theta) \ . \tag{2.22}$$

to within an arbitrary constant.

It should be pointed out that the asymptotic form of v_1 is actually

$$v_1 \sim A\ln\left(\frac{R}{r}\right) + F(r) + B_1(\theta) + o(1) \qquad \text{as } r \to 0 \tag{2.23}$$

where $F(r) = o(\ln r)$ does not contribute to the leading order terms in any of the governing equations, and therefore cannot be determined without a higher order analysis. (See for example Sirovich (1971) for a discussion of the Landau symbols 'O', and 'o'.) The function $F(r)$ has been neglected in this paper. If such a function were part of the correct asymptotic form of v_1 it would not affect the results presented here for the stresses, strain rates, or v_2. However, the results for the strains and v_1 would require important modification. That there is no similar function in the asymptotic form of v_2 follows from the argument above, which led to the exclusion of a logarithmic term from v_2.

The material strain rates are obtained from (2.5) and (2.20) as

$$\begin{aligned}\dot{\epsilon}_{11} &= -\frac{u}{2x_2}\sin 2\theta B_2'(\theta) \\ \dot{\epsilon}_{22} &= -\dot{\epsilon}_{11} \\ \dot{\epsilon}_{12} &= \frac{u}{2x_2}\left(\cos 2\theta B_2'(\theta) - A\right) \ .\end{aligned} \tag{2.24}$$

The strain rate components are therefore singular like $1/r$ as $r \to 0$.

In any region of the body that is at yield, the yield condition (2.14) can be satisfied identically if the two deviatoric stress components appearing in (2.14) are expressed in terms of a stress function $\psi(\theta)$ by

$$s_{22} = k\cos\left(2\theta - \psi(\theta)\right) \ , \qquad s_{12} = -k\sin\left(2\theta - \psi(\theta)\right) \ . \tag{2.25}$$

The deviatoric stress components are defined in terms of this particular function $\psi(\theta)$ in order to keep the notation similar to that used by Gao and Nemat-Nasser (1983). Because the hydrostatic stress σ is bounded (given the assumptions of the Appendix), it is taken to be a function of θ where

$$\sigma(\theta) = \frac{1}{2}(\sigma_{11} + \sigma_{22}) \tag{2.26}$$

In terms of the functions $B_2(\theta)$, $\psi(\theta)$, and $\sigma(\theta)$, the symmetry conditions (2.11) and the boundary conditions (2.12) are

$$\psi(0) = 0 \ , \quad B_2(0) = 0 \tag{2.27a}$$

and

$$\psi(\pi) = \pi \ , \quad \sigma(\pi) = k \tag{2.27b}$$

respectively, for the range $0 \leq \theta \leq \pi$. The system of ordinary differential equations that these functions must satisfy is obtained next.

The governing equations that remain to be satisfied are the two momentum balance equations (2.4) and the two rate equations (2.9) describing the material model. If the

representations (2.20), (2.25), and (2.26) are substituted into these four equations then a lengthy but straightforward manipulation of the results leads to

$$
\begin{aligned}
&(\psi'(\theta)-2)\left(\cos^2\psi(\theta)-m^2\sin^2\theta\right)=\frac{m^2}{k}A\cos(\psi(\theta)-2\theta)\\
&\frac{2kr}{u}\lambda(\theta)=k(\psi'(\theta)-2)\sin\theta\tan\psi(\theta)+2A\cos\theta\sec\psi(\theta)\\
&\sigma'(\theta)=k(\psi'(\theta)-2)\sin\psi(\theta)\\
&B_2'(\theta)=\frac{k}{m^2}(\psi'(\theta)-2)\cos\psi(\theta)\\
&B_1'(\theta)\tan\theta=B_2'(\theta)-A
\end{aligned}
\tag{2.28}
$$

These differential equations are considered for the case (2.17*b*), that is $A=0$, in the following section. It is shown in section 4 that it is not possible to construct a solution for $A\neq 0$ if the principle of maximum plastic work is to be enforced.

3. Solution for $A=0$

A solution to the system of equations (2.28) is sought for the case when $A=0$. This condition is equivalent to the assumption that the particle velocity components have the asymptotic form (2.17*b*). A solution of this form has been suggested by Lam and Freund (1985). Attention is first focused on the function $\psi(\theta)$ which, according to (2.28), must satisfy

$$(\psi'(\theta)-2)\left(\cos^2\psi(\theta)-m^2\sin^2\theta\right)=0 \tag{3.1}$$

in the interval $0\leq\theta\leq\pi$, varying from $\psi(0)=0$ to $\psi(\pi)=\pi$. Clearly (3.1) is satisfied if either $\psi'(\theta)=2$ or $\cos\psi(\theta)=\pm m\sin\theta$. The curves defined by $\cos\psi(\theta)=\pm m\sin\theta$ are shown in Fig. 2. Any line in this plane with a slope of 2 is an integral curve of $\psi'(\theta)=2$.

A complete solution is constructed in the following way. The curve $\psi(\theta)=2\theta$ satisfies the boundary condition at $\theta=0$, and the curve $\psi(\theta)=2\theta-\pi$ satisfies the boundary condition at $\theta=\pi$. The two curves can be joined by either of $\cos\psi(\theta)=\pm m\sin\theta$, and the choice is governed by the requirement that the plastic flow factor

$\lambda(\theta)$ given by (2.28) must be non-negative. The factor $(\psi'(\theta) - 2)$ is negative along both possible joining curves. Furthermore, $\tan\psi(\theta)\sin\theta = \pm\frac{1}{m}\sin\psi(\theta)$ where $\frac{1}{m}\sin\psi$ is non-negative along either curve. Therefore, the curve $\cos\psi(\theta) = -m\sin\theta$ must be chosen to render λ non-negative. For the case $m = \frac{1}{2}$, the complete solution for $\psi(\theta)$ is shown as the solid line in Fig. 2, and it is the only continuous solution satisfying the boundary conditions.

The interpretation of this solution is straight forward. The crack tip field consists of a uniform region in $0 \le \theta \le \theta_1^*$, a non-uniform region in $\theta_1^* \le \theta \le \theta_2^*$, and another uniform region in $\theta_2^* \le \theta \le \pi$. That the upstream and downstream regions are uniform follows directly from $\psi'(\theta) - 2 = 0$ and (2.25). The transition angles are determined to be

$$\theta_1^* = \sin^{-1}\left(\frac{m + \sqrt{8+m^2}}{4}\right) \; , \quad \theta_2^* = \pi - \sin^{-1}\left(\frac{-m + \sqrt{8+m^2}}{4}\right) \tag{3.2}$$

and the deviatoric stress components are determined by means of (2.25) from

$$\psi(\theta) = \begin{cases} 2\theta \; , & \text{if } 0 \le \theta \le \theta_1^* \\ \cos^{-1}(-m\sin\theta) \; , & \text{if } \theta_1^* \le \theta \le \theta_2^* \\ 2\theta - \pi \; , & \text{if } \theta_2^* \le \theta \le \pi \end{cases} \tag{3.3}$$

With $\psi(\theta)$ determined the remaining differential equations in (2.28) may integrated. The first integrals are constant in $0 \le \theta \le \theta_1^*$ and in $\theta_2^* \le \theta \le \pi$. In the intermediate region, after application of the boundary conditions, it is found that

$$\begin{aligned} B_1(\theta) &= \frac{k}{m^2}\big(\beta^2 F(\theta;m) - E(\theta;m) + 2m\sin\theta \\ &\qquad -\beta^2 F(\theta_1^*;m) + E(\theta_1^*;m) - 2m\sin\theta_1^*\big) + B_1^\circ \\ B_2(\theta) &= \frac{k}{m^2}\left(\sqrt{1-m^2\sin\theta} - 2m\cos\theta - \sqrt{1-m^2\sin\theta_1^*} + 2m\cos\theta_1^*\right) \\ \sigma(\theta) &= k\left(1 + m\sin\theta - 2E(\theta;m) - m\sin\theta_2^* + 2E(\theta_2^*;m)\right) \end{aligned} \tag{3.4}$$

where $F(\theta;m)$ and $E(\theta;m)$ are elliptic integrals of the first and second kind, respectively, both with parameter m, and $\beta^2 = 1 - m^2$. B_1° is a constant whose value cannot be determined from the asymptotic analysis.

With the functions $B_i(\theta)$ determined by (3.4), the expressions for strain rate components in terms of $B_i(\theta)$ (2.24), may be integrated along a line $x_2 = constant$ from $x_1 = x_2 \cot\theta_1^*$ through decreasing values of x_1 to determine the strain distribution in the non-uniform region $\theta_1^* \le \theta \le \theta_2^*$. The results of this integration, after application of the boundary conditions, are

$$
\begin{aligned}
\epsilon_{11}(\theta) &= -\epsilon_{22}(\theta) = -B_1(\theta) \\
\epsilon_{12}(\theta) &= \frac{k}{m^2}\left[\frac{m^2}{2}\ln\left(\frac{1-\sqrt{1-m^2\sin\theta}}{m\sin\theta}\right) + \sqrt{1-m^2\sin\theta} - 2m\cos\theta\right. \\
&\quad - \frac{m^2}{2}\ln\left(\frac{1-\sqrt{1-m^2\sin\theta_1^*}}{m\sin\theta_1^*}\right) - \sqrt{1-m^2\sin\theta_1^*} + 2m\cos\theta_1^* \\
&\quad \left. -m\ln\left(\tan\frac{\theta}{2}\right) + m\ln\left(\tan\frac{\theta_1^*}{2}\right)\right] .
\end{aligned} \tag{3.5}
$$

The strain components are uniform in $0 \le \theta \le \theta_1^*$, and in $\theta_2^* \le \theta \le \pi$. The uniform values may be obtained from (3.5).

4. Discussion of Solution with $A \neq 0$

In this section, the possibility of constructing an asymptotic solution with logarithmic dependence of v_1 on the radial coordinate r is considered. If the constant A, in (2.20a), is less than zero the crack is closing. It is therefore assumed that A is positive.

With reference to (2.27–8), the stress function $\psi(\theta)$ must satisfy the non-linear ordinary differential equation

$$
(\psi'(\theta) - 2)\left(\cos^2\psi(\theta) - m^2\sin^2\theta\right) = \frac{m^2}{k}A\cos(\psi(\theta) - 2\theta) , \tag{4.1}
$$

with boundary conditions

$$
\psi(0) = 0 \quad , \quad \psi(\pi) = \pi . \tag{4.2}
$$

The plastic flow factor $\lambda(\theta)$ may be manipulated into the form

$$\frac{2kr}{u}\lambda(\theta) = A\left(\frac{2\cos\theta\cos\psi(\theta) + m^2\sin\theta\sin(\psi(\theta) - 2\theta)}{\cos^2\psi(\theta) - m^2\sin^2\theta}\right) . \tag{4.3}$$

Recall that the conditions (4.1–3) correspond to a stress state satisfying the yield condition over the full angular range around the crack tip. This boundary value problem for $\psi(\theta)$ has been considered by Gao and Nemat-Nasser (1983), who noted that a continuous solution for $\psi(\theta)$ is inadmissible on the grounds that it would have to pass through a range of θ for which the plastic work rate is negative. This fact may be demonstrated in the following manner.

From (4.3) it is noted that the sign of the flow parameter $\lambda(\theta)$ depends on the relative signs of the functions $\lambda_1(\theta)$ and $\lambda_2(\theta)$ where

$$\begin{aligned} \lambda_1(\theta) &= 2\cos\theta\cos\psi + m^2\sin\theta\sin(\psi - 2\theta) , \\ \lambda_2(\theta) &= \cos^2\psi - m^2\sin^2\theta . \end{aligned} \tag{4.4}$$

The curves $\lambda_1(\theta) = 0$, denoted by (I,I′), and the curves $\lambda_2(\theta) = 0$, denoted by (II,II′) are shown in Fig. 3; the algebraic sign of the plastic flow parameter is indicated in each region.

Consider (4.1) and the first boundary condition (4.2). The solution curve $\psi(\theta)$ is well defined up to its point of intersection c with the curve $\cos\psi = m\sin\theta$ (curve II in Fig. 4), at which point $\psi' = \infty$. Equation (4.1) can be used to show that

$$\psi'(\theta) > 2 , \qquad \psi''(\theta) > 0 \tag{4.5}$$

for $0 \leq \theta < \theta_c$, where θ_c is the value of θ at the point c. This indicates that the solution of (4.1) up to the point c is of the form shown by curve III in Fig. 4, where the line $\psi = 2\theta$ (curve IV in Fig. 4) is shown for comparison. In particular, (4.5) shows that the solution curve must lie above the line $\psi = 2\theta$, at least until point $\theta = \theta_c$, where it encounters the curve $\cos\psi = m\sin\theta$ (curve II in Fig. 4) with $\psi'(\theta) = \infty$. The point c will always have the property that $\psi(\theta_c) < \pi/2$ and, therefore, the segment of the solution curve III must always lie inside the triangular region Odb defined by the lines

$\theta = 0$, $\psi = \pi/2$, and $\psi = 2\theta$. Point b in Fig. 4 is the intersection of line IV with $\psi = \pi/2$.

The only way for the solution to be extended continuously as a plastic state beyond the point c is through a region of negative plastic work rate, which is inadmissible. Gao and Nemat-Nasser (1983) proposed a remedy for this difficulty in the form of a discontinuity in $\psi(\theta)$ initiating at the point c. However, this approach is shown later in this section to be inconsistent with the principle of maximum plastic work, so that discontinuities in field variables are inadmissible within the context of the present mechanical problem.

Seemingly, the only way to construct a *continuous* asymptotic solution with logarithmic dependence of the velocity components on r and satisfying the maximum plastic work principle would involve the insertion of an elastically deforming sector starting at some point on the curve III. This possibility is considered subsequently in section 4, where it is shown that an elastic sector cannot possibly initiate at any point inside the region Odb, and therefore on curve III. The consequence is that a solution with logarithmic behavior in the velocity components is not admissible, and the velocity components must therefore be bounded.

Discontinuities

Possible discontinuities in field variables near the crack tip are considered next. For quasistatically moving surfaces of strong discontinuity in elastic-plastic solids, Drugan and Rice (1984) have shown that all stresses are continuous, although certain velocity components in the plane of the discontinuity may suffer jumps. The following analysis shows that for the dynamic problem considered in this paper all field quantities must be continuous.

Consider a hypothetical line discontinuity Σ propagating with the crack and inclined at an angle θ_o to the direction of crack growth. Define a local rectangular coordinate system x, y such that y is along the line of discontinuity; see Fig. 5. Let the jump in a quantity f be denoted by $[\![f]\!] \equiv f^+ - f^-$, where the superscript plus and minus indicate evaluation of f on the upstream and downstream sides of the discontinuity, respectively. Application of momentum balance across a moving surface of

discontinuity gives rise to the following well known jump conditions

$$[\![\sigma_{ij}]\!]n_j + m^2 \frac{c_d}{u}[\![v_i]\!] = 0 \ , \tag{4.6}$$

where n_j is the normal to the surface of discontinuity, and c_d is the speed of the surface of discontinuity in the direction of the normal. From Fig. 5 $c_d = u \sin\theta_o$, and (4.6) may be rewritten as

$$[\![\sigma_{xx}]\!] + m^2 \sin\theta_o [\![v_x]\!] = 0 \ , \qquad [\![\sigma_{xy}]\!] + m^2 \sin\theta_o [\![v_y]\!] = 0 \ . \tag{4.7}$$

In terms of the particle displacement components u_x and u_y, the velocity components are

$$v_x = \left(\sin\theta_o \frac{\partial u_y}{\partial y} - \cos\theta_o \frac{\partial u_x}{\partial y}\right) \ , \qquad v_y = -\left(\sin\theta_o \frac{\partial u_y}{\partial x} + \cos\theta_o \frac{\partial u_y}{\partial y}\right) \ , \tag{4.8}$$

where use has been made of the incompressibility condition $\partial u_x/\partial x + \partial u_y/\partial y = 0$.

Continuity of displacements implies, from Hadamard's lemma, that

$$\left[\!\!\left[\frac{\partial u_x}{\partial y}\right]\!\!\right] = \left[\!\!\left[\frac{\partial u_y}{\partial y}\right]\!\!\right] = 0 \ . \tag{4.9}$$

Using (4.9) in (4.8) shows that

$$[\![v_x]\!] = 0 \ , \qquad [\![v_y]\!] = -\sin\theta_o \left[\!\!\left[\frac{\partial u_y}{\partial x}\right]\!\!\right] = -2\sin\theta_o [\![\epsilon_{xy}]\!] \ . \tag{4.10}$$

In view of (4.10), equations (4.7) become

$$[\![\sigma_{xx}]\!] = 0 \ , \qquad [\![\sigma_{xy}]\!] = 2m^2 \sin^2\theta_o [\![\epsilon_{xy}]\!] \ . \tag{4.11}$$

Equations (2.6–7) combined with (4.11) imply that

$$[\![\sigma_{xx}]\!] = 0 \ , \qquad [\![\sigma_{yy}]\!] = -4[\![\epsilon^p_{yy}]\!] \ , \qquad [\![\sigma_{xy}]\!] = 2q[\![\epsilon^p_{xy}]\!] \ , \tag{4.12}$$

where $q = (m^2 \sin^2\theta_o)/(1 - m^2 \sin^2\theta_o)$.

It is now assumed that material particles follow a continuous path in strain space through the discontinuity and, furthermore, that if the jump in a quantity is zero, then

that quantity remains constant along the strain path through the discontinuity. In the following discussion the phrase "*inside the jump*" is understood to mean "*along the strain path from* ϵ_{ij}^{+} *to* ϵ_{ij}^{-} *within the discontinuity*". Thus, (4.12) may be written in the differential form

$$d\sigma_{xx} = 0 \ , \qquad d\sigma_{yy} = -4d\epsilon_{yy}^{p} \ , \qquad d\sigma_{xy} = 2qd\epsilon_{xy}^{p} \ , \tag{4.13}$$

inside the jump.

The maximum plastic work inequality is

$$(\sigma_{ij} - \sigma_{ij}^{*})d\epsilon_{ij}^{p} \geq 0 \ , \tag{4.14}$$

where σ_{ij}^{*} is any stress state on or inside the yield surface in stress space. Following the analysis by Drugan and Rice (1984) for the quasi-static case, the maximum plastic work inequality is integrated across the discontinuity to obtain

$$W = \int_{\epsilon_{ij}^{p+}}^{\epsilon_{ij}^{p-}} (\sigma_{ij} - \sigma_{ij}^{*})d\epsilon_{ij}^{p} \geq 0 \ . \tag{4.15}$$

W is now written in the form $W \equiv W^{p} - W^{*}$ where

$$W^{p} = \int_{\epsilon_{ij}^{p+}}^{\epsilon_{ij}^{p-}} \sigma_{ij}d\epsilon_{ij}^{p} \ , \qquad \text{and } W^{*} = \int_{\epsilon_{ij}^{p+}}^{\epsilon_{ij}^{p-}} \sigma_{ij}^{*}d\epsilon_{ij}^{p} \ , \tag{4.16}$$

Use of (4.13) allows evaluation of W^{p} as

$$\begin{aligned} W^{p} &= \sigma_{xx}^{\Sigma} \int_{\epsilon_{xx}^{p+}}^{\epsilon_{xx}^{p-}} d\epsilon_{xx}^{p} + \frac{1}{q} \int_{\sigma_{xy}^{+}}^{\sigma_{xy}^{-}} \sigma_{xy}d\sigma_{xy} - \frac{1}{4} \int_{\sigma_{yy}^{+}}^{\sigma_{yy}^{-}} \sigma_{yy}d\sigma_{yy} \ , \\ &= -\sigma_{xx}^{\Sigma}[\![\epsilon_{xx}^{p}]\!] - \frac{1}{2q}\left(\sigma_{xy}^{+} + \sigma_{xy}^{-}\right)[\![\sigma_{xy}]\!] + \frac{1}{8}\left(\sigma_{yy}^{+} + \sigma_{yy}^{-}\right)[\![\sigma_{yy}]\!] \ , \end{aligned} \tag{4.17}$$

where the superscript Σ indicates the evaluation of a continuous field quantity at the discontinuity. Use of (4.12) in (4.17) results in

$$W^{p} = -\frac{1}{2}(\sigma_{ij}^{+} + \sigma_{ij}^{-})[\![\epsilon_{ij}^{p}]\!] \ . \tag{4.18}$$

The integral W^* is easily evaluated to give

$$W^* = -\sigma^*_{ij}[\![\epsilon^p_{ij}]\!] \ . \tag{4.19}$$

Finally, combining (4.18) and (4.19) shows that W is given by

$$W = \frac{1}{2}(2\sigma^*_{ij} - \sigma^+_{ij} - \sigma^-_{ij})[\![\epsilon^p_{ij}]\!] \geq 0 \ . \tag{4.20}$$

Because the material is non-hardening, any stress state in the body may be chosen for σ^*_{ij}. In particular, the choices $\sigma^*_{ij} = \sigma^+_{ij}$, σ^-_{ij}, or any stress state inside the jump are valid. Choosing $\sigma^*_{ij} = \sigma^+_{ij}$ and σ^-_{ij} in (4.20) gives $[\![\sigma_{ij}]\!][\![\epsilon^p_{ij}]\!] \geq 0$ and $[\![\sigma_{ij}]\!][\![\epsilon^p_{ij}]\!] \leq 0$ respectively. Clearly these relations are satisfied if and only if

$$[\![s_{ij}]\!][\![\epsilon^p_{ij}]\!] = 0 \tag{4.21}$$

where use has been made of $\epsilon^p_{kk} = 0$. Using (4.12) with (4.21) implies that

$$[\![s_{xy}]\!] = \pm\sqrt{q}\,[\![s_{yy}]\!] \ . \tag{4.22}$$

As noted above, any state inside the jump is a possible candidate for σ^*_{ij} in equation (4.14), and therefore (4.22) must hold for any pair of states inside the jump. This implies that

$$s_{xy} = \pm\sqrt{q}\,s_{yy} + K \tag{4.23}$$

inside the jump, where K is an unknown constant. In addition, (4.12) shows that no part of the strain path through the discontinuity may be purely elastic, and hence the yield condition must be satisfied at every state inside the jump.

Use of (4.23) in the yield condition (2.14) shows that

$$[\![\sigma_{yy}]\!] = [\![\sigma_{xy}]\!] = 0 \ . \tag{4.24}$$

These results, together with (4.10–13), show that all field quantities must be continuous across the discontinuity, that is,

$$[\![\sigma_{ij}]\!] = 0 \ , \qquad [\![v_i]\!] = 0 \ . \tag{4.25}$$

In particular, (4.25) requires that $[\![s_{ij}]\!] = 0$, and therefore that $\psi(\theta)$ is continuous, $[\![\psi(\theta)]\!] = 0$.

<u>Elastic Sectors</u>

In this section the possibility of including an elastic sector in the asymptotic field, initiating at some angle θ^* on curve III of Fig. 4, is considered. In an elastic sector $\dot{\epsilon}^p_{ij} = 0$ and, consequently, the stress rate and strain rate are related by

$$\dot{\epsilon}_{ij} = \frac{1}{2}\dot{s}_{ij} \ . \tag{4.27}$$

Use of (2.3), (2.19), and (4.27) in the momentum equations (2.4) shows that the velocity potential $\phi(r,\theta)$ must satisfy

$$\nabla^2(\hat{\nabla}^2\phi) = 0 \qquad \left(\hat{\nabla}^2 = \beta^2\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) \ . \tag{4.28}$$

(Recall that $\beta^2 = 1 - m^2$.) As can be seen from (2.22), the velocity potential has the general asymptotic form

$$\phi(r,\theta) = rg(\theta) - Ar\sin\theta\ln r \ . \tag{4.29}$$

Representation (4.29) implies that

$$\hat{\nabla}^2\phi = \frac{G(\theta)}{r} \ , \tag{4.30}$$

where

$$G(\theta) = (1 - m^2\sin^2\theta)\left(g(\theta) + g''(\theta) - 2A\cos(\theta)\right) \ . \tag{4.31}$$

Using (4.30) in (4.28) shows that $G(\theta)$ satisfies the ordinary differential equation $G''(\theta) + G(\theta) = 0$ which, with (4.31), shows that $g(\theta)$ satisfies the inhomogeneous equation

$$g''(\theta) + g(\theta) = \frac{a_o\sin\theta + b_o\cos\theta}{1 - m^2\sin^2\theta} + 2A\cos\theta \tag{4.32}$$

where a_o and b_o are constants to be determined. The general solution of this equation has the form

$$g(\theta) = a_i g_i(\theta) \ , \qquad i = 1\text{–}5 \qquad \text{(sum implied)} \tag{4.33}$$

where the a_i are undetermined constants, and $a_3 = -A$. The functions $g_i(\theta)$ are given by

$$g_1(\theta) = \cos\theta \ , \quad g_2(\theta) = \sin\theta \ , \quad g_3(\theta) = \theta\cos\theta \ ,$$

$$g_4(\theta) = \frac{1}{2}\cos\theta \ln(1 - m^2\sin^2\theta) + \sin\theta\left(\theta - \beta\tan^{-1}(\beta\tan\theta)\right) \ , \tag{4.34}$$

$$g_5(\theta) = -\frac{1}{2}\sin\theta\ln(1 - m^2\sin^2\theta) + \cos\theta\left(\theta - \frac{1}{\beta}\tan^{-1}(\beta\tan\theta)\right) \ .$$

The strains may be obtained from the velocity potential (4.29) by means of (2.5) and (2.19) by integration along $x_2 = constant$. The shear strain ϵ_{12} can only be determined to within an arbitrary function of x_2. This unknown function is determined by the boundary conditions at $\theta = \theta^*$. Expressions for the deviatoric stresses are obtained through integration of the constitutive equations (4.27),

$$s_{ij} = 2\left(\epsilon_{ij} - \epsilon^p_{ij}(x_2)\right) \ , \tag{4.35}$$

and the hydrostatic stress σ through integration of the momentum equations (2.4). Although the expression for ϵ_{12} contains an unknown function of x_2, it turns out, through use of the momentum equations, that the difference $\epsilon_{12} - \epsilon^p_{12}$ in (4.35) must be a constant multiplied by x_2. This difference is asymptotically negligible, and the result is that s_{12} can be determined. The resulting expressions for the deviatoric and hydrostatic stresses are

$$s_{11} = -2a_2 + 2a_4\left(\beta\tan^{-1}\theta(\beta\tan\theta) - \theta\right) + a_5\ln(1 - m^2\sin^2\theta)$$
$$+ 2A(1 + \ln r) - 2\epsilon^p_{11}(x_2) \ , \tag{4.36a}$$

$$s_{12} = s_o + a_4\left(m^2\ln r + \frac{1}{2}(2 - m^2)\ln(1 - m^2\sin^2\theta)\right)$$
$$+ a_5\left(2\theta - (\beta + \frac{1}{\beta})\tan^{-1}(\beta\tan\theta)\right) - 2A\theta \ , \tag{4.36b}$$

$$\sigma = \sigma_o - m^2 a_4\theta + m^2(A - a_5)\ln r - 2\epsilon^p_{11}(x_2) \ , \tag{4.36c}$$

where s_o and σ_o are constants.

By means of (4.36a), boundedness of s_{11} in the elastic sector requires that

$$\epsilon_{11}^{p}(x_2) \sim -A\ln(r\sin\theta) + \epsilon_o \ , \tag{4.37}$$

where ϵ_o is a constant. Also, because s_{12} must be bounded, (4.36b) shows that the constant $a_4 = 0$. Continuity of the hydrostatic stress at an elastic-plastic boundary implies that σ is bounded in elastic regions, and therefore $a_5 = -A(2-m^2)/m^2$. The resulting expressions for the stress components are

$$s_{11} = c_1 - A\left(2\ln(\sin\theta) + \frac{2-m^2}{m^2}\ln(1-m^2\sin^2\theta)\right) \ , \tag{4.38a}$$

$$s_{12} = c_2 - \frac{A}{m^2}\left(4\theta - \frac{2-m^2}{\beta}\tan^{-1}(\beta\tan\theta)\right) \ , \tag{4.38b}$$

$$\sigma = c_3 - 2A\ln(\sin\theta) \ , \tag{4.38c}$$

where c_1, c_2, and c_3 are constants.

With reference to Fig. 4, the possibility of an elastic sector initiating on curve III at some angle $\theta = \theta^*$, with $0 \le \theta^* \le \theta_c$, is considered. The purpose here is to establish whether or not it is possible to avoid the region of negative plastic work at $\theta = \theta_c$ through the introduction of a region of elastic behavior. In any such elastic sector the unloading condition

$$\frac{d}{d\theta}\left(s_{22}^2 + s_{12}^2\right) \le 0 \tag{4.39}$$

must hold in some neighborhood of θ^*, and in particular at the point $\theta = \theta^*$. Hence the inequality

$$s_{22}^{e}(\theta^*)s_{22}^{e\prime}(\theta^*) + s_{12}^{e}(\theta^*)s_{12}^{e\prime}(\theta^*) \le 0 \tag{4.40}$$

must hold, where the superscript e indicates that quantities are evaluated on the elastic side of the elastic-plastic boundary.

From continuity of stresses at an elastic-plastic boundary, (2.25) implies

$$s_{22}^{e}(\theta^*) = k\cos(2\theta^* - \psi^*) \ , \qquad s_{12}^{e}(\theta^*) = -k\sin(2\theta^* - \psi^*) \tag{4.41}$$

where $\psi^* \equiv \psi(\theta^*)$. Use of (4.38$a$), (4.38$b$), and (4.41) in the inequality (4.40) shows

that ψ^* must satisfy

$$\tan(\psi^* - 2\theta^*) \leq \frac{2\cot\theta^* \cos 2\theta^*}{4\cos^2\theta^* - m^2} \ . \tag{4.42}$$

It can be shown that this inequality is not satisfied anywhere inside the triangle Odb of Fig. 4. But, as was mentioned earlier in this section, the point (ψ^*, θ^*) must lie on curve III, which in turn must lie inside triangle Obd. Hence an elastic sector cannot be inserted into the plastic field to circumvent the region of negative plastic work. This implies that there is no solution curve if the region of negative plastic work at $\theta = \theta_c$, indicated in Figs. 3–4, is present. Therefore, ways to eliminate it must be sought. Inspection of the governing ordinary differential equations (2.28) reveals that this can only be accomplished by choosing $A = 0$, and hence the velocity component v_1 may not have a logarithmic singularity in r.

For $A = 0$, it can be seen from (4.38) that the only possible elastic sector is a region of constant stress which must be at yield in light of full continuity at an elastic-plastic boundary. But such constant state regions are included in the solution presented in section 3, and it is therefore concluded that the solution in section 3 is unique within the solution class considered.

5. Discussion

There are several reasons for the study of the asymptotic crack tip field for dynamic growth of a mode I crack in an elastic-plastic material. To begin with, numerical methods, which provide the only means for obtaining full field solutions within this problem class, are of limited reliability very close to the crack edge. The ability to match computed fields to asymptotic fields in some sense establishes confidence in the numerical results. This point was discussed by Lam and Freund (1985). Secondly, the influence of material inertia on the distribution of stress and deformation near the crack edge is of interest in assessing mechanisms of crack advance because these fields represent the environment in which the mechanisms are operative. Thirdly, in the situation considered here, more than one solution has been suggested in the literature. Thus, an additional objective here has been to identify a discriminating feature among the proposed asymptotic solutions.

The angular variations of the stress components for nondimensional crack speeds $m = 0.1$, and 0.9 are shown in Fig. 6. The influence of inertia, aside from increasing the transition angles θ_1^* and θ_2^* somewhat, is merely to cause a moderate reduction of the hydrostatic stress σ in the upstream region. Of course, this implies that both σ_{11} and σ_{22} are similarly reduced. Figure 7 shows the variation of the strain components for $m = 0.1$, 0.5, and 0.9. Inertial effects are more significant here than for stress; the downstream value of ϵ_{12} is reduced almost to the upstream level, and the variation of both ϵ_{12} and ϵ_{22} is reduced in the transition region. This results in very little strain variation for large m. Note that the unknown constant B_1^o has been chosen to be $B_1^o = 1$ for displaying ϵ_{22}. B_1^o can only be determined by matching this solution to the outer solution. However, it might expected that $B_1^o \to 0$ as $m \to 1$.

The solution presented here includes no elastic unloading in any angular sector about the crack tip, and therefore it cannot reduce to the corresponding quasistatic solution of Rice, Drugan, and Sham (1980). A somewhat similar situation arises in mode III, where the dynamic steady-state solution fails to reduce to the corresponding quasistatic solution. However, this issue was resolved by Douglas and Freund (1982) when they demonstrated that the domain of validity of the dynamic asymptotic solution vanishes with vanishing crack tip speed. Carrying this observation over to the mode I problem, it appears that here too the domain of validity of the dynamic asymptotic field vanishes with vanishing crack tip speed. As pointed out by Lam and Freund (1985), the solution presented here is an extension of the Achenbach and Dunayevsky (1981) solution to the full crack speed range $0 < m < 1$ for elastic incompressibility. The stress components reduce to those observed in the Prandtl punch field as $m \to 0$, as noted by Achenbach and Dunayevsky (1981). The components of strain (elastic strain is bounded) become unbounded as $m \to 0$, although ϵ_{12} remains zero in the upstream sector. The region where ϵ_{22} is singular as $m \to 0$ depends on the behavior of B_1^o as $m \to 0$.

Concerning the previously proposed asymptotic solutions, it has been noted that the main difference arises from whether or not discontinuities in stress and/or particle velocity components are admitted across lines emanating from the crack tip. As is shown in the foregoing discussion, if it is required that the strain histories must abide by the principle of maximum plastic work through the jump, then such discontinuities must

be ruled out. Consequently, only asymptotic solutions that exhibit continuous angular variations in stress and particle velocity are in full compliance with the maximum plastic work principle.

The question of whether or not the strain paths through the jumps *should* be required to satisfy the principle of maximum plastic work could be raised, of course. Within the context of gas dynamics, Courant and Friedrichs (1948) show that, for a mechanical analysis of the propagation of discontinuities, the sequence of mechanical states experienced by a material particle as a discontinuity propagates across it must be the same as if the transition from the initial state to the final state has occurred in a smooth simple wave. The reason for this outcome, in a somewhat simplified form, is that in a mechanical description of the phenomenon (as opposed to a more complete thermodynamic description) the behavior of the material does not have the flexibility afforded by thermodynamic properties to depart from the mechanical constitutive law imposed. These observations may be carried over to the present case of elastic-plastic crack growth. The model is strictly mechanical so it may be expected that the behavior of the material in a jump discontinuity must also be representable as the limit of a smooth wave as the slope of the smooth wave becomes indefinitely steep. If this is so, then it follows immediately that the sequence of states in the jump must be consistent with the principle of maximum plastic work.

It should be pointed out that Courant and Friedrichs (1948) only state that for a mechanical description of the transition through a discontinuity, the initial and final states are the same as if the transition had occurred in a simple wave. However, their proof of this statement also demonstrates that the sequence of states in the discontinuity must also be the same as if the transition had occurred in a simple wave. In order to describe the state of a material particle passing through the discontinuity, they used essentially the same assumption that was used by Drugan and Rice (1984), and that was used in section 4 of this paper, namely, that if the jump in some quantity is zero, then that quantity remains constant along the path through the discontinuity.

Acknowledgement

We are grateful to Professor W. Drugan of the University of Wisconsin for helpful discussions on this work. The research support of the Office of Naval Research, Mechanics Division through contract N00014–85–K–0597 and of the NSF Materials Research Laboratory at Brown University, grant DMR 83–16893 is gratefully acknowledged.

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Appendix: Bounded Hydrostatic Stress

In this section it is shown that under certain assumptions on the behavior of the field variables and their first two derivatives with respect to r, the hydrostatic stress σ is bounded at the crack tip. To this end, the momentum equations (2.4) may be referred to crack tip polar coordinates as

$$\begin{aligned} \frac{\partial\sigma}{\partial r} &= \frac{m^2}{r}\frac{\partial v_2}{\partial\theta} - \frac{\partial s_{rr}}{\partial r} - \frac{1}{r}\frac{\partial s_{r\theta}}{\partial\theta} - 2\frac{s_{rr}}{r} \,, \\ \frac{1}{r}\frac{\partial\sigma}{\partial\theta} &= -m^2\frac{\partial v_2}{\partial r} - \frac{1}{r}\frac{\partial s_{\theta\theta}}{\partial\theta} - \frac{\partial s_{r\theta}}{\partial r} - 2\frac{s_{r\theta}}{r} \,, \end{aligned} \tag{A.1}$$

where (2.3) and the incompressibility condition have been used to write

$$\omega_r = \frac{u^2}{r}\frac{\partial v_2}{\partial\theta} \qquad \omega_\theta = -u^2\frac{\partial v_2}{\partial r} \,, \tag{A.2}$$

and ω_r and ω_θ are the radial and circumferential components, respectively, of the particle accelerations. Differentiation of the second of equations (A.1) with respect to r gives

$$m^2\frac{\partial^2 v_2}{\partial r^2} = \frac{1}{r^2}\left(\frac{\partial\sigma}{\partial\theta} - \frac{\partial s_{\theta\theta}}{\partial\theta} + 2s_{r\theta}\right) - \frac{1}{r}\left(\frac{\partial^2\sigma}{\partial r\partial\theta} - \frac{\partial^2 s_{\theta\theta}}{\partial r\partial\theta} + 2\frac{\partial s_{r\theta}}{\partial r}\right) + \frac{\partial^2 s_{r\theta}}{\partial r^2} \,, \tag{A.3}$$

while differentiation and manipulation of the momentum equations in rectangular coordinates yields

$$m^2\nabla^2 v_2 = -\frac{\partial^2 s_{12}}{\partial x_1^2} + \frac{\partial^2 s_{12}}{\partial x_2^2} - 2\frac{\partial^2 s_{22}}{\partial x_1\partial x_2} \,. \tag{A.4}$$

It is assumed that all functions and their first and second derivatives have limits as $r \to 0$, and further that all functions $g(r,\theta)$ have the property that $\partial g(r,\theta)/\partial\theta = \mathrm{O}\left(g(r,\theta)\right)$ (See for example Sirovich (1971) for a discussion of the Landau symbols 'O', and 'o'.) Then (A.1), (A.3) and (A.4) may be replaced by

$$\frac{\partial\sigma}{\partial r} = \frac{m^2}{r}\frac{\partial v_2}{\partial\theta} + \mathrm{O}\left(\frac{1}{r}\right) \,, \qquad \frac{1}{r}\frac{\partial\sigma}{\partial\theta} = -m^2\frac{\partial v_2}{\partial r} + \mathrm{O}\left(\frac{1}{r}\right) \,, \tag{A.5}$$

$$m^2\frac{\partial^2 v_2}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\sigma}{\partial\theta}\right) + \mathrm{O}\left(\frac{1}{r^2}\right) \,, \tag{A.6}$$

and

$$\nabla^2 v_2 = \mathrm{O}\left(\frac{1}{r^2}\right) \,, \tag{A.7}$$

respectively. A solution is now sought of the form

$$\sigma(r,\theta)=f(r)g(\theta)+h(r,\theta)\ , \qquad \text{as } r\to 0 \tag{A.8}$$

where $f(r)$ is singular as $r\to 0$ and $h(r,\theta)=\mathrm{o}(f(r))$ for any θ. Substituting $(A.8)$ into the second of equations $(A.5)$ and integrating gives

$$m^2 v_2(r,\theta)=-g'(\theta)\int\frac{f(r)}{r}dr-\int\frac{1}{r}\frac{\partial h}{\partial\theta}dr+\mathrm{O}(\ln r)\ . \tag{A.9}$$

Sirovich (1971) presents a discussion of the conditions under which the order relations may be integrated. Substitution of $(A.9)$ into the first of equations $(A.5)$ and integrating once again gives

$$\sigma(r,\theta)=-g''(\theta)\int\left(\frac{1}{r}\int\frac{f(r)}{r}dr\right)dr-\int\left(\frac{1}{r}\int\frac{1}{r}\frac{\partial^2 h}{\partial\theta^2}dr\right)dr+\mathrm{O}\left(\ln^2 r\right)\ . \tag{A.10}$$

Comparison of $(A.8)$ with $(A.10)$ reveals that

$$\begin{aligned} g(\theta)f(r)=&-g''(\theta)\int\left(\frac{1}{r}\int\frac{f(r)}{r}dr\right)dr\\ &-\int\left(\frac{1}{r}\int\frac{1}{r}\frac{\partial^2 h}{\partial\theta^2}dr\right)dr+\mathrm{O}\left(\ln^2 r\right)+\mathrm{o}(f(r))\ . \end{aligned} \tag{A.11}$$

If $g''(\theta)\neq 0$, then from $(A.11)$ one of the following three conditions must hold;

(i)
$$\int\left(\frac{1}{r}\int\frac{f(r)}{r}dr\right)dr=\mathrm{O}\left(\int\left(\frac{1}{r}\int\frac{1}{r}\frac{\partial^2 h}{\partial\theta^2}dr\right)dr\right)\ ,$$

(ii)
$$\int\left(\frac{1}{r}\int\frac{f(r)}{r}dr\right)dr=\mathrm{O}\left(\ln^2 r\right)\ ,$$

(iii)
$$g(\theta)f(r)=-g''(\theta)\int\left(\frac{1}{r}\int\frac{f(r)}{r}dr\right)dr+\mathrm{o}(f(r))\ .$$

By previous assumptions $\partial^2 h(r,\theta)/\partial\theta^2=\mathrm{O}(h(r,\theta))=\mathrm{o}(f(r))$ and therefore condition (i) implies that $\int\left(1/r\int f(r)/r dr\right)dr=\mathrm{o}\left(\int\left(1/r\int f(r)/r dr\right)dr\right)$ which is clearly not possible. Furthermore, successive division by r and integration with respect to r of

the relation $1 = o(f(r))$ shows that $\ln^2 r = o(f(r)/r)$. Thus (ii) also implies that $\int (1/r \int f(r)/r dr)\, dr = o\left(\int (1/r \int f(r)/r dr)\, dr\right)$ which is impossible. Condition (iii) implies that $f(r) \sim \alpha \int (1/r \int f(r)/r dr)\, dr$, and $g''(\theta) + \alpha g(\theta) = 0$. Because neither condition (i) nor (ii) is possible it is concluded that either

$$g''(\theta) = 0 \ , \qquad \text{or} \qquad g''(\theta) + \alpha g(\theta) = 0 \ . \tag{A.12}$$

The solution for σ is now examined in the neighborhood of $\theta = \pi$. The constitutive equation (2.9) shows that

$$\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} = \frac{2\mu}{u} \lambda s_{12} - \frac{\partial s_{12}}{\partial x_1} \ . \tag{A.13}$$

Differentiation of $(A.13)$ with respect to x_1 and use of the incompressibility condition gives

$$\frac{\partial^2 v_2}{\partial x_1^2} - \frac{\partial^2 v_2}{\partial x_2^2} = \frac{\partial}{\partial x_1}\left(2\mu\lambda s_{12} - u\frac{\partial s_{12}}{\partial x_1}\right) \ . \tag{A.14}$$

Combining $(A.14)$ with $(A.7)$ shows that

$$\frac{\partial^2 v_2}{\partial x_1^2} = \frac{\partial}{\partial x_1}\left(\mu\lambda s_{12} - \frac{u}{2}\frac{\partial s_{12}}{\partial x_1}\right) + O\left(\frac{1}{r^2}\right) \ . \tag{A.15}$$

The traction free crack face condition requires $s_{12}(r,\pi) = 0$. This implies, in conjunction with $(A.15)$, that

$$\left.\frac{\partial^2 v_2}{\partial r^2}\right|_{\theta=\pi} = O\left(\frac{1}{r^2}\right) \ . \tag{A.16}$$

Substitution of $(A.8)$ into $(A.6)$ shows that

$$m^2\frac{\partial^2 v_2}{\partial r^2} = -g'(\theta)\left(\frac{f(r)}{r}\right)' - \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial h}{\partial \theta}\right) + O\left(\frac{1}{r}\right) \ . \tag{A.17}$$

If it is assumed that

$$\left(\frac{f(r)}{r}\right)' = O\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial h}{\partial \theta}\right)\right) \ , \tag{A.18}$$

then integration and multiplication by r implies that $f(r) = o(f(r))$ which is clearly not possible. Similarly, the assumption that $(f(r)/r)' = O(1/r^2)$ leads to the conclusion

that $f(r) = \mathrm{O}(1)$, which contradicts the assumption that $f(r)$ is singular. Therefore $(A.17)$ may be rewritten as

$$m^2 \frac{\partial^2 v_2}{\partial r^2} = -g'(\theta) \left(\frac{f(r)}{r} \right)' + \mathrm{o} \left(\left(\frac{f(r)}{r} \right)' \right) . \qquad (A.19)$$

Comparing $(A.16)$ to $(A.19)$ shows that

$$g'(\pi) \left(\frac{f(r)}{r} \right)' = \mathrm{O} \left(\frac{1}{r^2} \right) + \mathrm{o} \left(\left(\frac{f(r)}{r} \right)' \right) . \qquad (A.20)$$

Integration of $(A.20)$ with respect to r and multiplication by r shows that either $f(r) = \mathrm{O}(1)$ or $g'(\pi) = 0$. Because $f(r)$ is singular it must be that $g'(\pi) = 0$. In addition, the traction free crack face condition implies that $\sigma(r, \pi) = -s_{22}(r, \pi)$. The deviatoric stresses are bounded for all r, and therefore $(A.8)$ implies that $g(\pi) = 0$. Thus, if $f(r)$ is singular then $g(\theta)$ must satisfy $(A.12)$ with the boundary conditions

$$g(\pi) = 0 \ , \qquad \text{and} \qquad g'(\pi) = 0 \ . \qquad (A.21)$$

The only possible solution to either of equations $(A.12)$, which satisfies $(A.21)$, is $g(\theta) \equiv 0$. Thus, if $f(r)$ is singular then $g(\theta) \equiv 0$, which implies that $\sigma(r, \theta)$ may not be singular in the region near $\theta = \pi$ if it can be represented by $(A.8)$. This result is independent of whether the material is deforming elastically or plastically in that region. In addition, it was shown in section 4 that the hydrostatic stress must be continuous throughout the body. Therefore σ must be bounded in all angular sectors about the crack tip.

<u>Figure Captions</u>

1. Diagram of crack tip with rectangular and polar coordinate systems. Both coordinate systems move with the crack tip in the x_1 direction with constant speed u.

2. Solution curves for $\psi(\theta)$ of the type $\cos\psi(\theta) = \pm\sin\theta$ for $m = 0.1$, 0.5, and 0.9. The solid line represents the complete solution for $m = 0.5$ and $A = 0$.

3. Diagram of the regions in which the plastic flow factor λ is positive (+), and negative (−), for $m = 0.5$ and $A \neq 0$.

4. Curves which delineate regions where the plastic flow factor λ is either negative or positive (see Figure 3), and a possible solution curve III. The curve III was obtained by numerically integrating the governing equation for $\psi(\theta)$, for $m = 0.5$ and $A = 1$. Curve IV is the line $\psi = 2\theta$.

5. Discontinuity oriented at θ_d with respect to the crack tip, with local x, y rectangular coordinate system. The upstream (+) and downstream (−) regions are as indicated.

6. Variation of certain stress measures for $0 \leq \theta \leq \pi$. All stress quantities have been normalized by the elastic shear modulus μ.

7. Variation of strain components for $0 \leq \theta \leq \pi$. Note that the upstream value of ϵ_{11} has been taken to be 1 for all crack speeds.

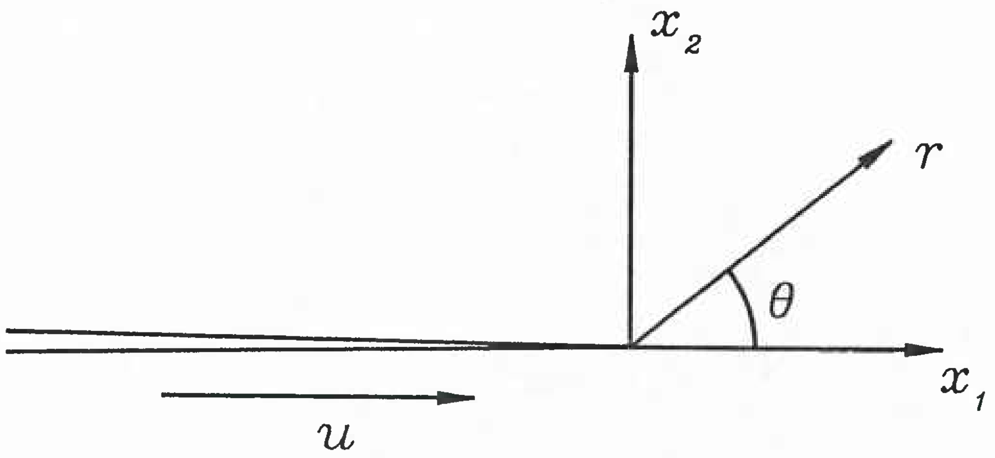
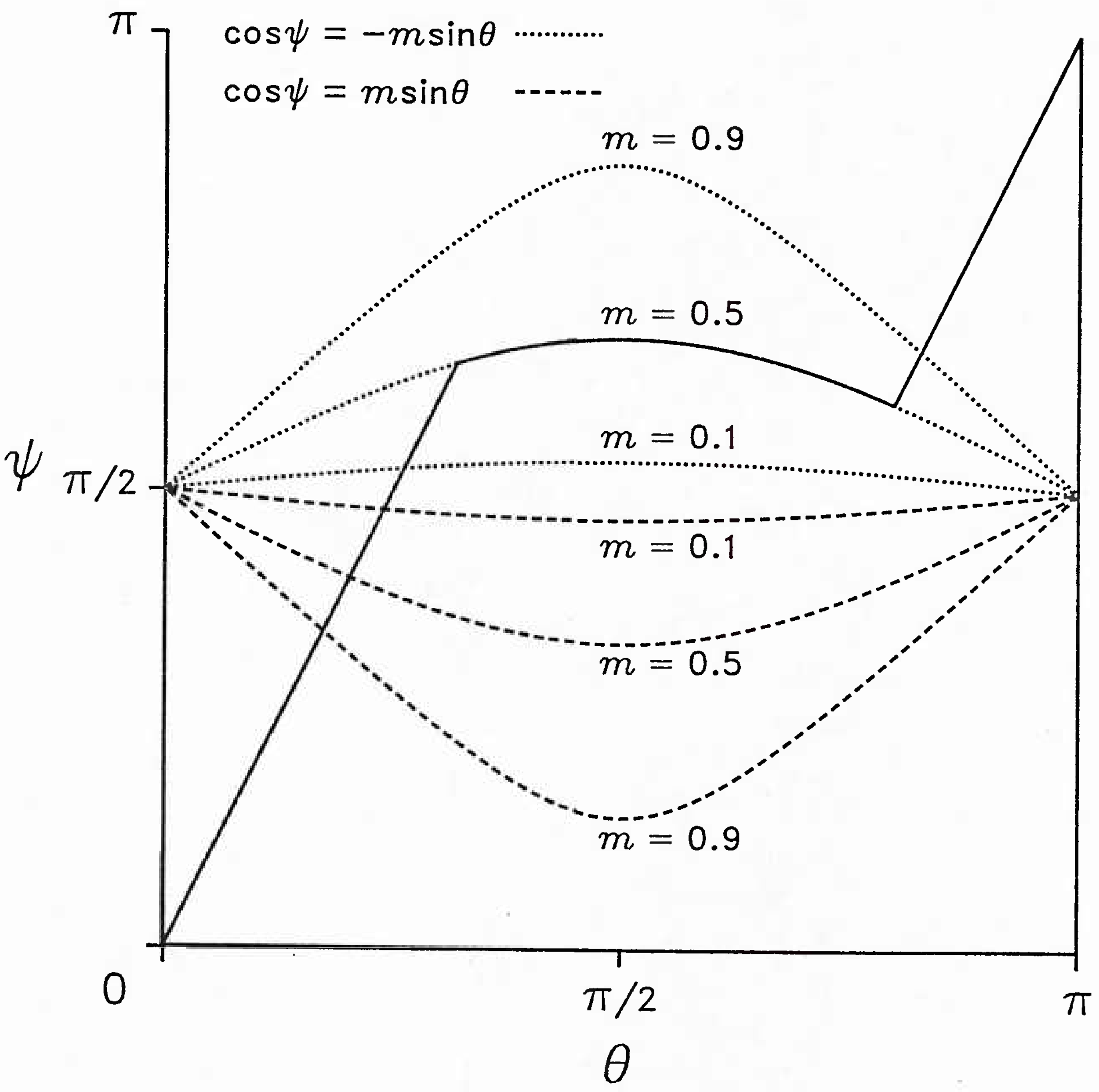


Figure 1



π
$\cos\psi = -m\sin\theta$
$\cos\psi = m\sin\theta$
$m = 0.9$
$m = 0.5$
$m = 0.1$
ψ
$\pi/2$
$m = 0.1$
$m = 0.5$
$m = 0.9$
0
$\pi/2$
π
θ

Figure 2

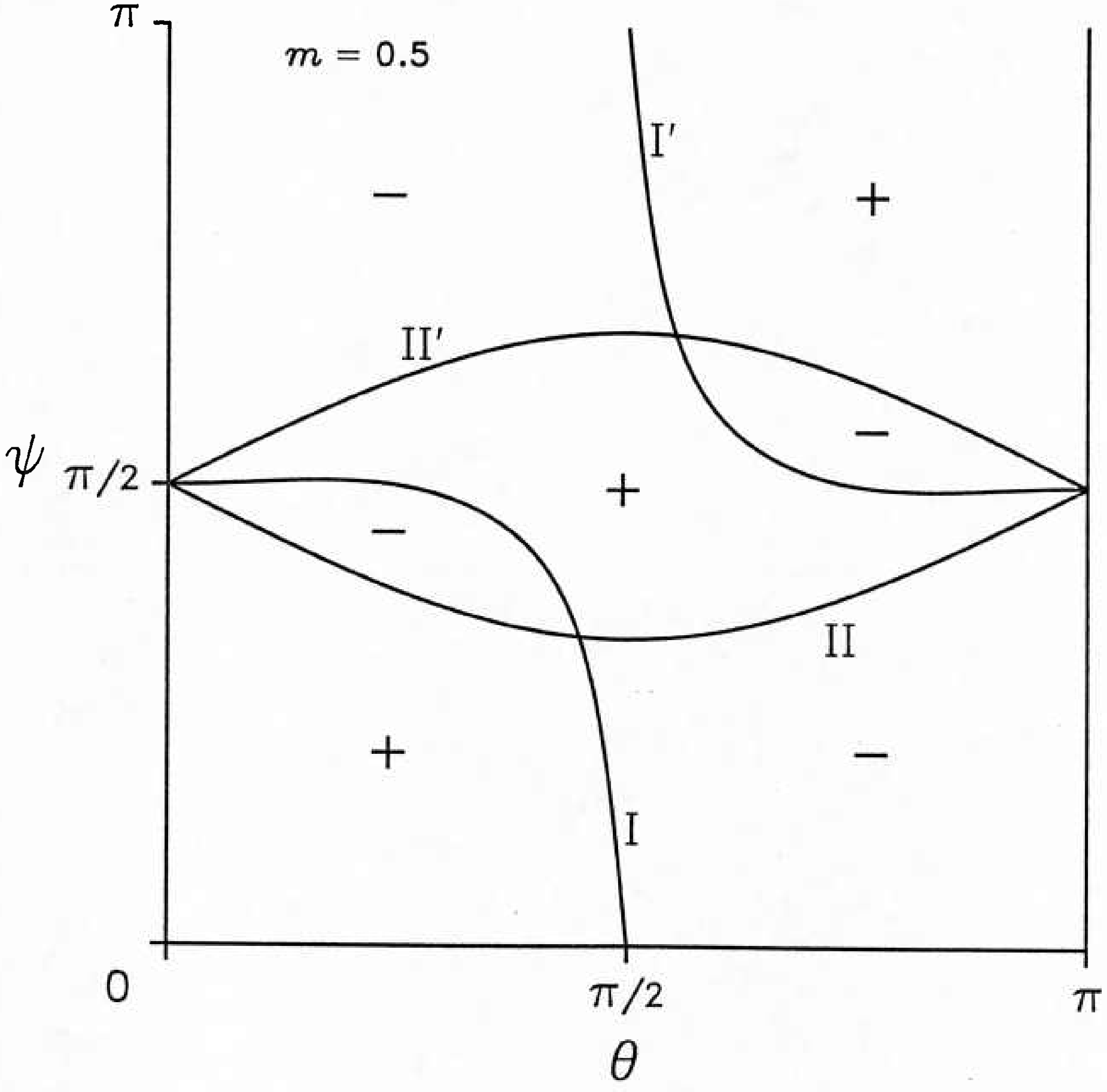


Figure 3

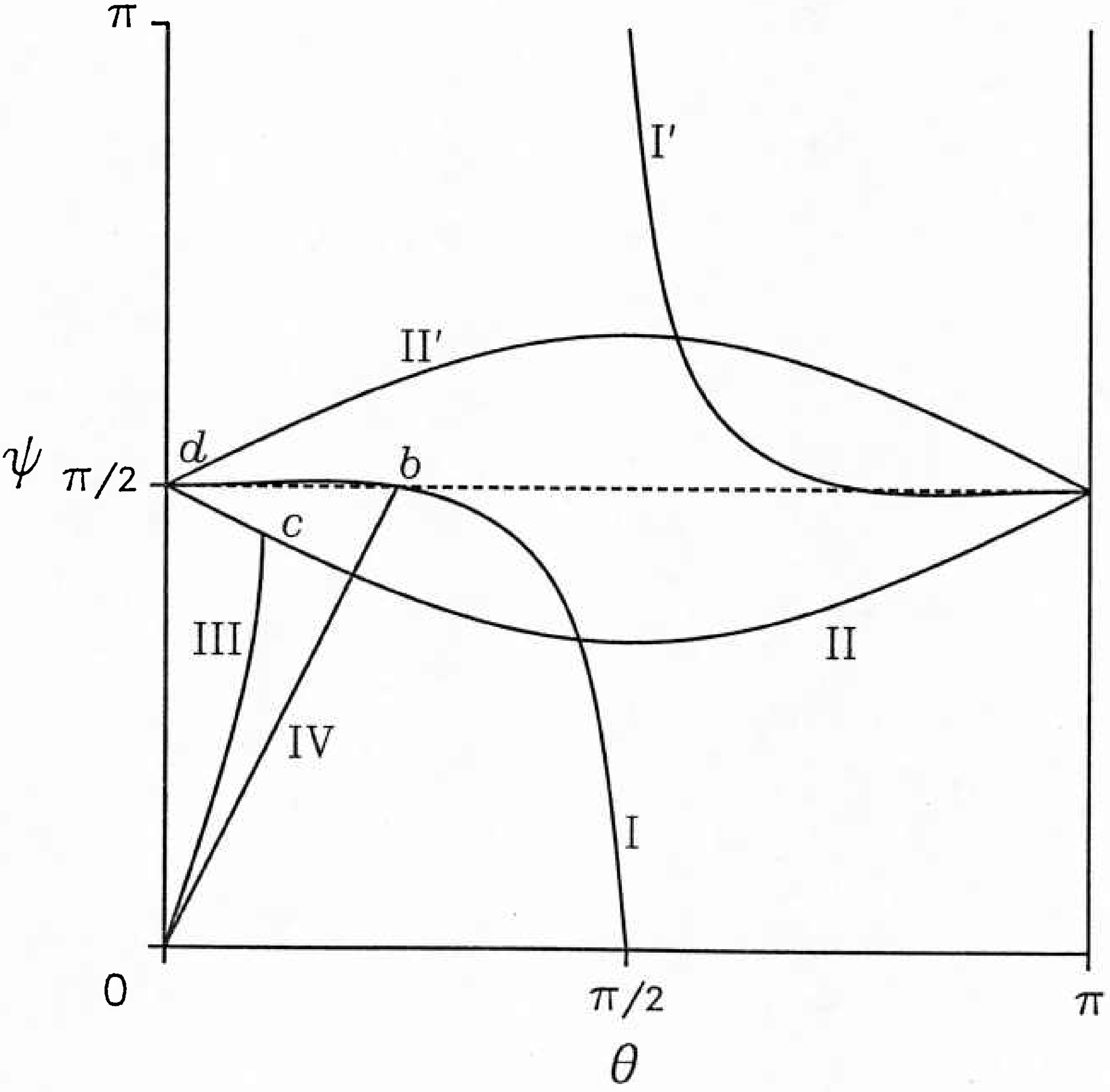


Figure 4

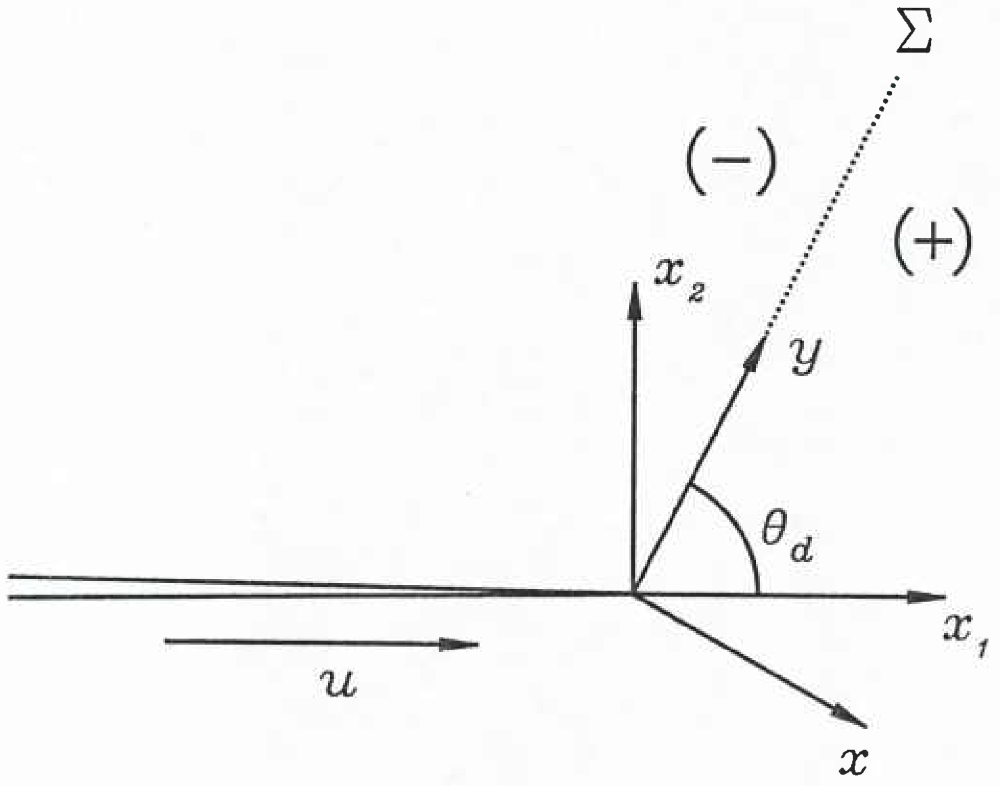


Figure 5

Stresses

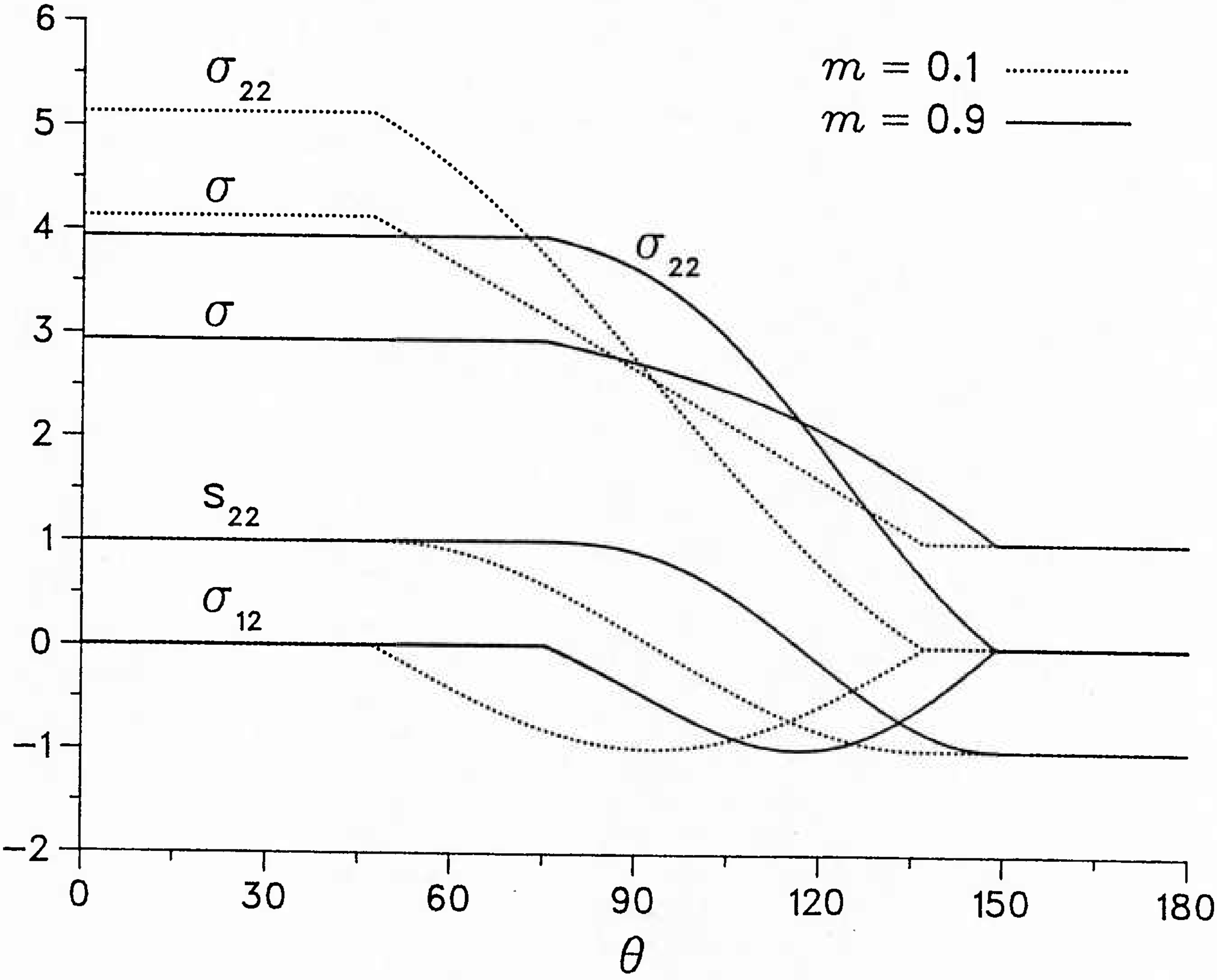


Figure 6

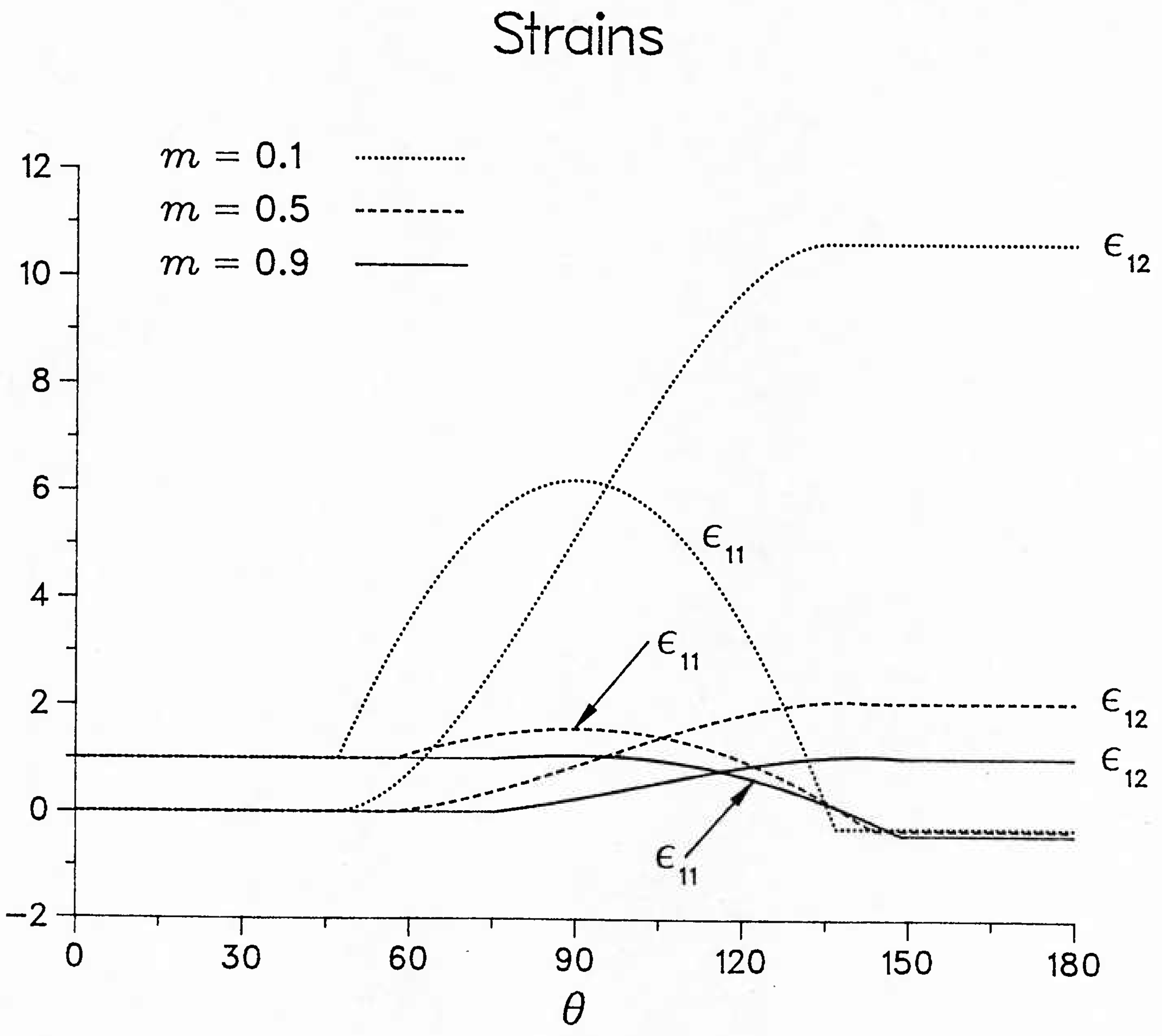

Strains
$m = 0.1$
$m = 0.5$
$m = 0.9$
12
10
8
6
4
2
0
−2
0
30
60
90
120
150
180
θ
ϵ_{12}
ϵ_{11}
ϵ_{11}
ϵ_{12}
ϵ_{12}
ϵ_{11}

Figure 7